Reduced order methods for optimal flow control: FEniCS-based applications

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FEniCS 2021

Motivations

Starting Point

Reduced Order Methods (ROMs) for parameterized Optimal Flow Control Problems (OFCPs) in environmental sciences

PDEs-based

Several simulations for different values of physical and/or geometrical parameters

(Uncertainty Quantification, Parameter Estimation...)

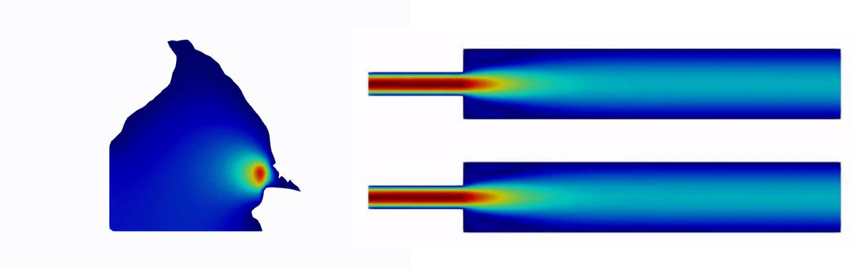
OFCPs

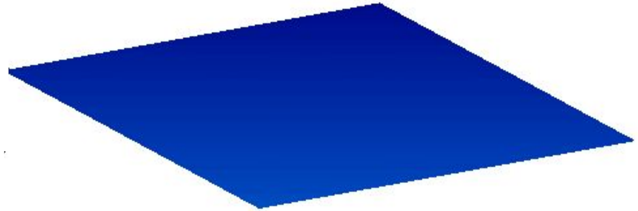
DATA-based

Scattered

expensive and difficult to collect

complex to interpret





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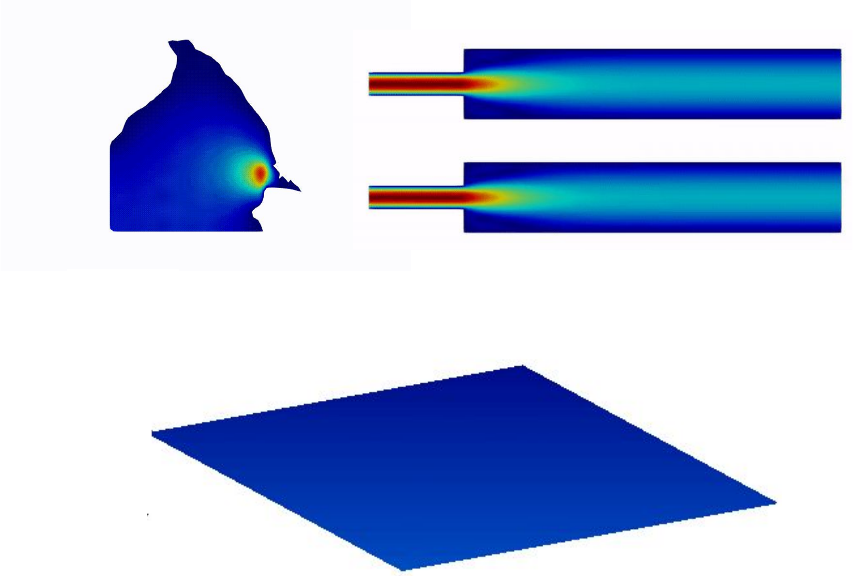

The Optimal Control Pipeline

PDEs
Data μ
Control

Cost
Functional
Minimization

Optimal
Solution

Solution
Interpretation

μ changes

ROM framework
time-dependent nonlinear problems
(POD, RB)

General Formulation
easy to adapt to several PDEs

Advanced Applications
(UQ, Bifurcations)

multiphenics RBniCS

https://mathlab.sissa.it/multiphenics
https://www.rbnicsproject.org/

Outline:

1. Problem Formulation
2. Ideas behind ROMs
3. Advanced Applications
    - Shallow Waters Equations (SWE)
    - Steering Bifurcations
    - Uncertainty Quantification (UQ)

Advisors:
Prof. Gianluigi Rozza (SISSA)
Dr. Francesco Ballarin (UNICATT)

Collaborators:
Prof. Rob Stevenson (UvA)
Giuseppe Carere (UvA)
Dr. Federico Pichi (SISSA/EPFL)

Problem Formulation

$$\min_{(y,u)\in\mathcal{Y}\times\mathcal{U}} J(y,u;\boldsymbol{\mu}) := \min_{(y,u)\in\mathcal{Y}\times\mathcal{U}} \frac{1}{2}\int_0^T \|y(\boldsymbol{\mu})-y_d(\boldsymbol{\mu})\|^2_{Y(\Omega_{\mathrm{OBS}})}\,dt + \int_0^T \frac{\alpha}{2}\|u(\boldsymbol{\mu})\|^2_{U(\Omega_u)}\,dt$$

such that $\boxed{\mathcal{E}(y,u;\boldsymbol{\mu}) = 0}$

Linear

Non-linear

Time-dependent

$$(\mathrm{DIM} = 3\mathcal{N}) \begin{cases} D_y\mathcal{L}((y,u,p);y_{\mathrm{d}},\boldsymbol{\mu})[\omega] = 0 & \forall \omega \in \mathbb{Y}, \\ D_u\mathcal{L}((y,u,p);y_{\mathrm{d}},\boldsymbol{\mu})[\kappa] = 0 & \forall \kappa \in \mathbb{U}, \\ D_p\mathcal{L}((y,u,p);y_{\mathrm{d}},\boldsymbol{\mu})[\zeta] = 0 & \forall \zeta \in \mathbb{Y}. \end{cases}$$

PDEs
Graetz Flows (temperature)
Advection Diffusion (pollutant)
Shallow Waters (coastal management)
Navier-Stokes (haemodynamics)

Controls
Boundary, Part of the domain, the whole domain, forcing terms…
Control **changes** the usual behaviour of the solution.

Optimal Control is a very powerful mathematical tool

- more reliable solutions
- great versatility

However it is **costly** (based on the solution of three equations)

Problem Formulation

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Optimal Control is a very powerful mathematical tool

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- great versatility

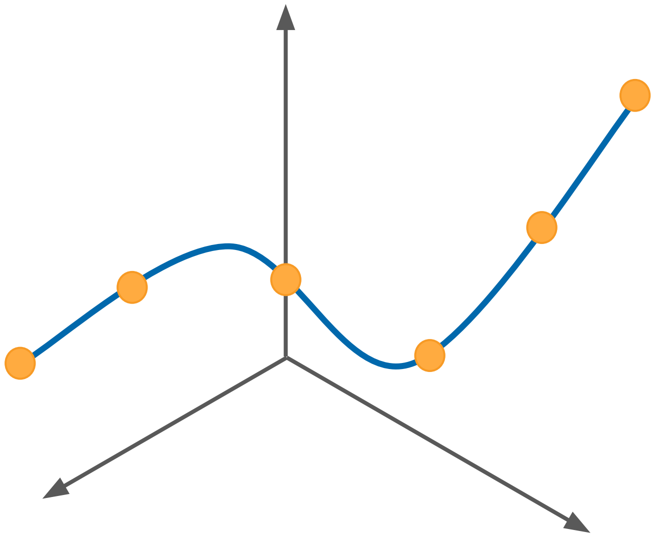
However it is costly (based on the solution of three equations) **most of all in a parametrized setting**

ROMs

Exploit the parametric structure of the problem

Ideas behind ROMs

Explore the solutions **snapshots** and how they change with respect to the parameter



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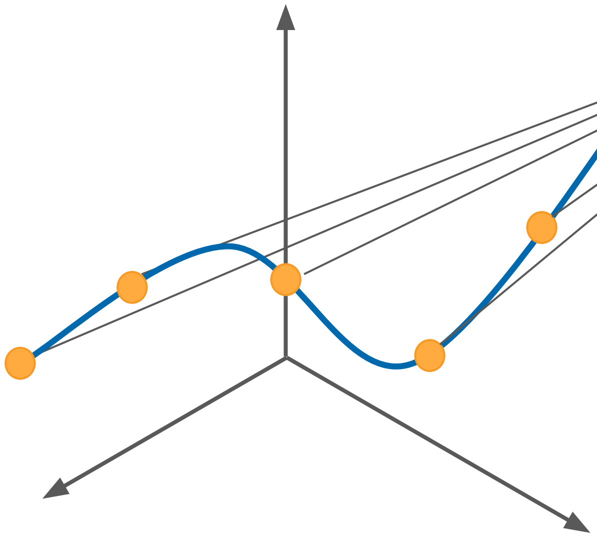

Building phase
POD (SVD, PCA)
RB (Adaptive technique)

$\longrightarrow$ Z^T

$$A_N = Z^T A Z$$

use μ A_N for each new parameter evaluation

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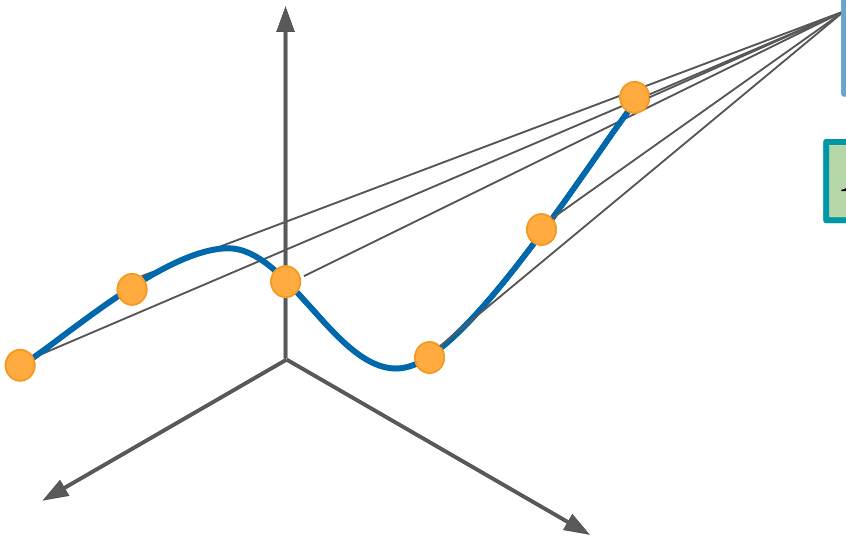

Building phase
POD (SVD, PCA)
RB (Adaptive technique)

$\rightarrow$ Z^T

$$A_N = Z^T \; A \; Z$$

Saving computational time

use μ A_N for each new parameter evaluation

How do you use RBniCS?

Example of Steady Coercive State Equation

```
class EllipticOptimalControl(EllipticOptimalControlProblem):

    # Default initialization of members
    def __init__(self, V, **kwargs):
        # Call the standard initialization
        EllipticOptimalControlProblem.__init__(self, V, **kwargs)
        # ... and also store FEniCS data structures for assembly
        assert "subdomains" in kwargs
        assert "boundaries" in kwargs
        self.subdomains, self.boundaries = kwargs["subdomains"], kwargs["boundaries"]
        yup = TrialFunction(V)
        (self.y, self.u, self.p) = split(yup)
        zvq = TestFunction(V)
        (self.z, self.v, self.q) = split(zvq)
        self.dx = Measure("dx")(subdomain_data=subdomains)
        self.ds = Measure("ds")(subdomain_data=boundaries)
        # Regularization coefficient
        self.alpha = 0.01
        # Desired state
        self.y_d = Constant(1.0)
        # Customize linear solver parameters
        self._linear_solver_parameters.update({
            "linear_solver": "mumps"
        })
```

In the class you define the parameters multiplied by the various forms defined as in FEniCS

Simple modifications for more complicated problems or to use other algorithms

```
# 1. Read the mesh for this problem
mesh = Mesh("data/mesh1.xml")
subdomains = MeshFunction("size_t", mesh, "data/mesh1_physical_region.xml")
boundaries = MeshFunction("size_t", mesh, "data/mesh1_facet_region.xml")

# 2. Create Finite Element space (Lagrange P1)
scalar_element = FiniteElement("Lagrange", mesh.ufl_cell(), 1)
element = MixedElement(scalar_element, scalar_element, scalar_element)
V = FunctionSpace(mesh, element, components=["y", "u", "p"])

# 3. Allocate an object of the EllipticOptimalControl class
elliptic_optimal_control = EllipticOptimalControl(V, subdomains=subdomains, boundaries=boundaries)
mu_range = [(1.0, 3.5), (0.5, 2.5)]
elliptic_optimal_control.set_mu_range(mu_range)

# 4. Prepare reduction with a reduced basis method
pod_galerkin_method = PODGalerkin(elliptic_optimal_control)
pod_galerkin_method.set_Nmax(10)

# 5. Perform the offline phase
lifting_mu = (1.0, 1.0)
elliptic_optimal_control.set_mu(lifting_mu)
pod_galerkin_method.initialize_training_set(100)
reduced_elliptic_optimal_control = pod_galerkin_method.offline()

# 6. Perform an online solve
online_mu = (2.0, 2.)
reduced_elliptic_optimal_control.set_mu(online_mu)
reduced_elliptic_optimal_control.solve()
reduced_elliptic_optimal_control.export_solution(filename="online_solution")
print("Reduced output for mu =", online_mu, "is", reduced_elliptic_optimal_control.compute_output())
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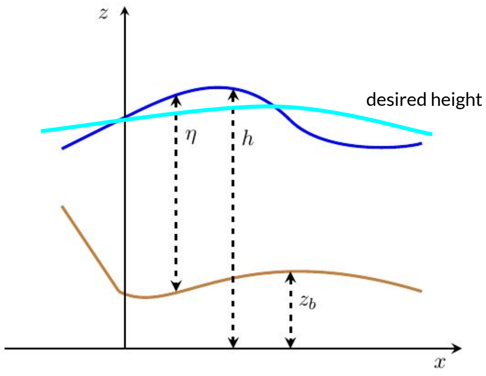
Coastal Height Management

Minimisation of

$$\frac{1}{2}\int_0^T \int_\Omega (h - h_d(\mu_3))^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega (\boldsymbol{v} - \boldsymbol{v}_d(\mu_3))^2 dxdt + \frac{\alpha}{2}\int_0^T \int_\Omega \boldsymbol{u}^2 dxdt$$

constrained to

$$\begin{aligned}
&\boldsymbol{v}_t + \mu_1 \Delta \boldsymbol{v} + \mu_2 (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + g\nabla\eta - \boldsymbol{u} = 0 && \text{in } \Omega \times [0,T],\\
&h_t + \operatorname{div}(\eta \boldsymbol{v}) = 0 && \text{in } \Omega \times [0,T],\\
&\boldsymbol{v} = \boldsymbol{v}_0 && \text{on } \Omega \times \{0\},\\
&h = h_0 && \text{on } \Omega \times \{0\},\\
&\boldsymbol{v} = \boldsymbol{0} && \text{on } \partial\Omega \times [0,T].
\end{aligned}$$

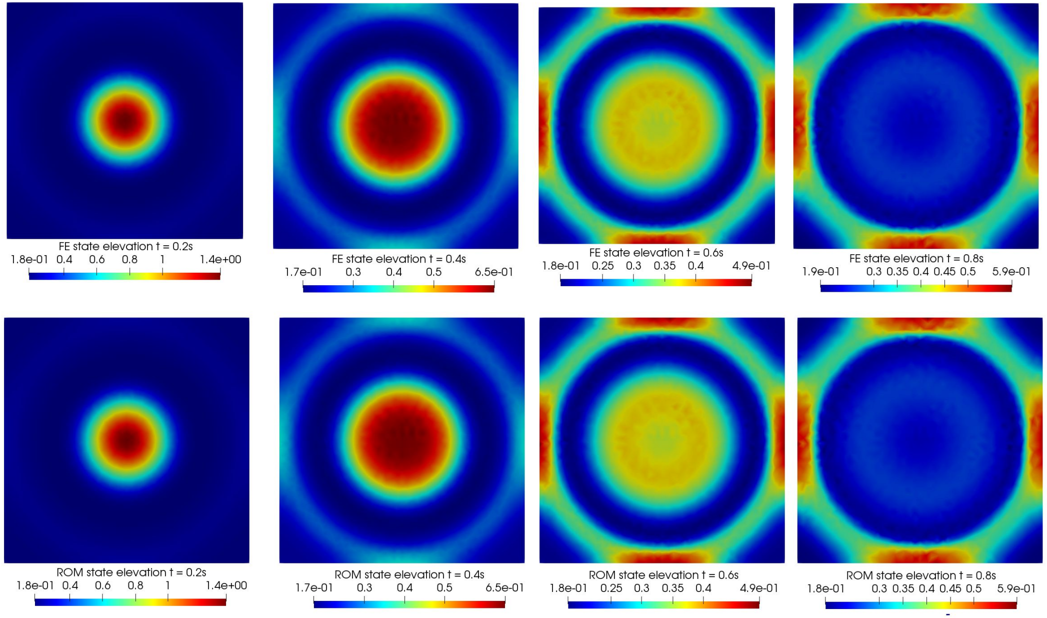


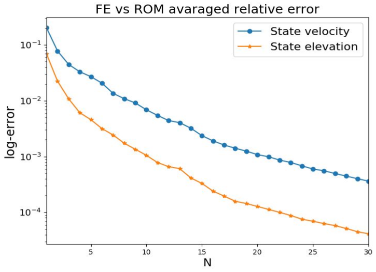
GOAL: recover parametrized desired height and velocity profiles with distributed control

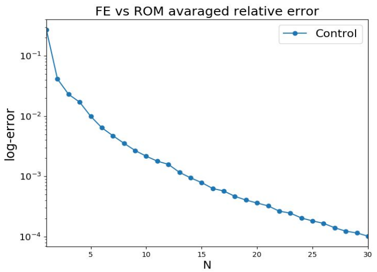
$$\begin{aligned}
[0,T] &= [0, 0.8],\\
\Omega &= [0,10]\times[0,10],\\
\alpha &= 10^{-5},\\
\Delta t &= 0.1.
\end{aligned}$$

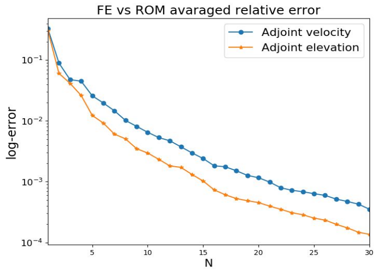
Straightforward implementation in RBniCS with standard modifications of Nonlinear problems classes

Coastal Height Management









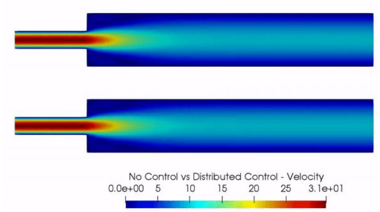
[Strazzullo, Ballarin, Rozza, POD-Galerkin Model Order Reduction for Parametrized Nonlinear Time Dependent Optimal Flow Control: an Application to Shallow Water Equations. *Submitted*, 2021]

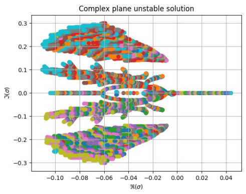
Errors ∼ 1e-4, Speedup ∼ 30
ROM vs FE dim = 270 vs 94’016.

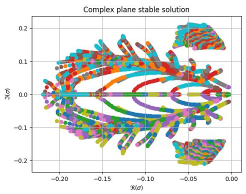
Advanced Applications

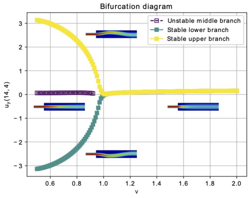
Optimal control + Bifurcations in Navier-Stokes:

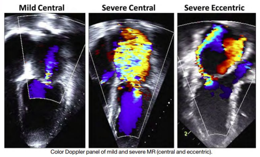
- Does optimal control affect the system?
- Does optimal control change stability?
- Does it change the stability analysis?
- What is the most “natural” configuration?



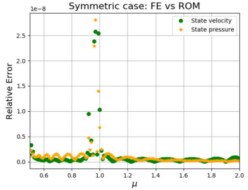






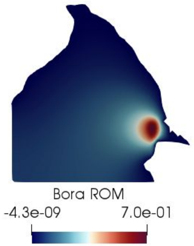


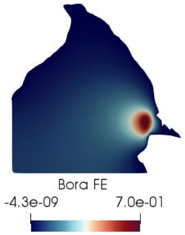
credits: American Society of Echocardiography Guideline Valvular Regurgitation

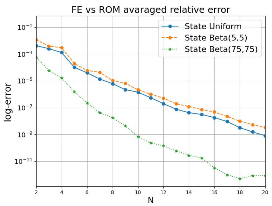


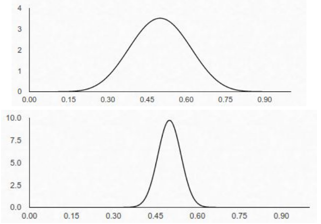
Optimal control + Uncertainty Quantification:

- Input-output relation?
- **Weighted**-POD
- More knowledge is better (less basis need)









[Pichi, Strazzullo, Ballarin, Rozza, Driving bifurcating parametrized nonlinear PDEs by optimal control strategies: application to Navier-Stokes equations with model order reduction. *Submitted, 2020*]

[Carere, Strazzullo, Ballarin, Rozza, Stevenson, Weighted POD-reduction for parametrized PDE-constrained Optimal Control Problems with random inputs and its applications to environmental sciences *Submitted, 2021*]

Conclusions

ROMs and OCPs

- Great mathematical tool to reach a desired configuration
- Expensive to solve
- RBniCS and multiphenics application to OFCPs
  - great versatility
  - simple to code

https://www.rbnicsproject.org/

Applications

- POD for nonlinear time-dependent problems (SWEs)
- OFCPs and bifurcations
- OFCPs and UQ

Acknowledgements

European Union Funding for Research and Innovation -- Horizon 2020 Program -- in the framework of European Research Council Executive Agency: Consolidator Grant H2020 ERC CoG 2015 AROMA-CFD project 681447 ``Advanced Reduced Order Methods with Applications in Computational Fluid Dynamics''.

Thank you for your attention!