Generating layer-adapted meshes using mesh PDEs

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IRISH RESEARCH COUNCIL
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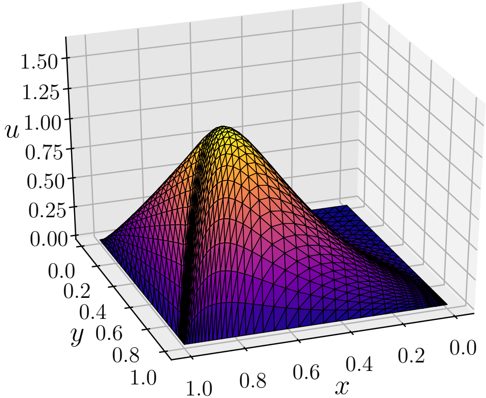
FEniCS PROJECT

Motivation

We want to solve singularly perturbed differential equations (SPDEs) whose solutions have boundary layers, so require special layer-adapted meshes.



Example with boundary layers near $x = 1$ and $y = 1$.

Motivation (cont.)

Our goal is to generate layer-adapted meshes for solving SPDEs, with mesh-adaption driven by Mesh Partial Differential Equations (MPDEs) [Huang and Russell, 2011].

Typically layer adapted meshes are formulated using *a priori* information about the SPDE's solution; the most successful (arguably) of these is due to Bakhvalov [Bakhvalov, 1969]. For problems in 2D, they are restricted to tensor product grids.

In this talk we will present:

1. a more general formulation based on MPDEs, and
2. an algorithm for efficiently solving these nonlinear problems.

Results and source code are available as: *Generating layer-adapted meshes using mesh partial differential equations*; `osf.io/dpexh/` (to appear in Numer. Math. Theor. Meth. Appl.) [Hill and Madden, 2021]

Reaction-diffusion equation and method

Our SPDEs are reaction-diffusion problems of the form

$$-\varepsilon^2 \Delta u + ru = f \text{ in } \Omega \subseteq \mathbb{R}^d, \text{ with } u|_{\partial\Omega} = 0, \qquad (1)$$

where $d = 1, 2$; $\varepsilon > 0$, r and f are given (smooth) functions and $r \geq \beta^2$ on $\overline{\Omega}$, with $\beta > 0$.

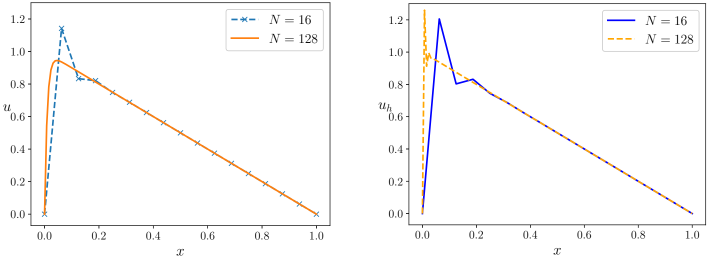
When ε is small the solutions exhibit boundary layers.

We use a standard Galerkin finite element method (FEM), with linear elements, to compute our numerical solutions to (1), and implement the method in FEniCS [Alnæs et al., 2015].

Solutions to (2) on **uniform** meshes

Model 1D scalar reaction-diffusion equation

$$-\varepsilon^2 u'' + u = 1 - x, \quad \text{on } (0,1), \quad u(0) = u(1) = 0. \tag{2}$$



(a) $\varepsilon = 10^{-2}$ (b) $\varepsilon = 10^{-4}$

The oscillations occur in the solutions when $\varepsilon < C/N$ due to the lack of stability in the discrete problem.

Layer-resolving meshes

We are interested in using layer-resolving meshes, which concentrate mesh points in regions where large variations occur in the solution. In 1D we consider these meshes in terms of this *"mesh generating function"*.

Definition

A mesh generating function (on [0,1]) is a strictly monotonic bijective function $\varphi : [0,1] \to [0,1]$ that maps a uniform mesh $\xi_i = i/N$, to a possibly non-uniform mesh $x_i = \varphi(i/N)$, for $i = 0, 1, \ldots, N$.

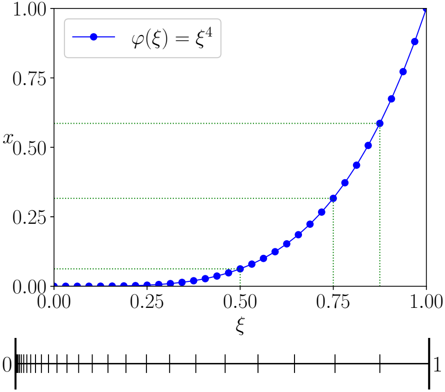


Figure: Mesh generated when $\varphi(\xi) = \xi^4$

Bakhvalov mesh (via equidistribution)

One method of generating a Bakhvalov mesh for (2) is by equidistributing the function [Linß, 2010],

$$\rho(x) = \max\left\{1, K\frac{\beta}{\varepsilon}\exp\left(-\frac{\beta x}{\sigma\varepsilon}\right)\right\}, \tag{3}$$

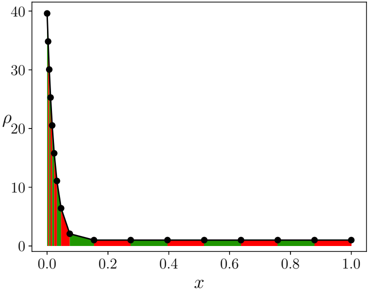
where σ, β and K are constants.

That is, one computes the mesh $\omega^N := \{0 = x_0, x_1, \ldots, x_N = 1\}$ such that

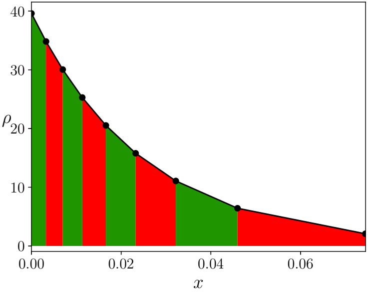
$$\int_{x_i}^{x_{i+1}} \rho(x)dx = \frac{1}{N}\int_0^1 \rho(x)dx \text{ for } i = 0, 1, ..., N-1.$$

This is a nonlinear problem, as is the classic method of generating a Bakhvalov mesh.

Equidistribution of ρ



(a) $x \in [0, 1]$



(b) zoomed

Figure: integral of ρ on the resulting mesh.

When $\varepsilon \ll 1$, ρ decays rapidly near $x = 0$. Therefore, we use the Gauss-Lobatto quadrature rule when solving the 1D MPDE. We thank Jørgen Dokken for pointing us towards a nice implementation.

Mesh PDE (MPDE)

A (moving) mesh PDE is presented in [Huang et al., 1994] as a way to generate specially adapted meshes:

1. A PDE whose solution is a mesh generating function is posed.
2. The PDE features a coefficient, ρ, that controls the concentration of points in the resulting mesh.
3. Classically, ρ depends on (local error estimates for) the solution.
4. However, we will use the basic idea to generate *a priori* Bakhvalov-style meshes.

From equidistribution to an MPDE

We derived the MPDE,

$$-\left(\rho(x)x(\xi)'\right)' = 0 \text{ for all } \xi \in (0,1), \quad x(0) = 0 \text{ and } x(1) = 1, \qquad (4)$$

using the equidistribution principle.

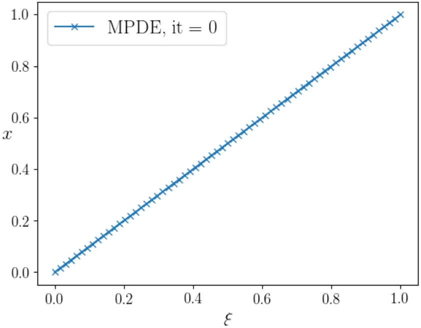
The BCs are necessary to result in a mesh generating function.

The solution to (4) is a Bakhvalov mesh when ρ is as defined in (3).

Since this is a nonlinear problem we use a fixed point iteration method to find a solution.

Solving the MPDE by an FEM

Example of how the mesh evolved when $N = 64$ and $\varepsilon = 10^{-3}$



Efficiency of the method

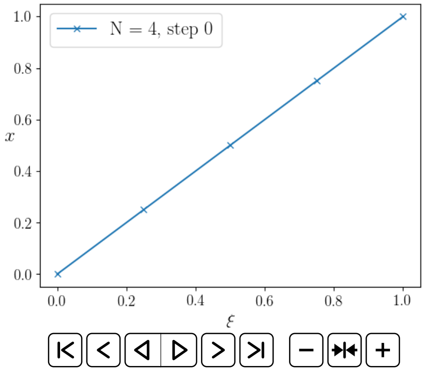
The number of iterations required was $\mathcal{O}(N)$. So, to improve the efficiency of the method, we

1. start with a mesh with 4 intervals,
2. apply 3 iterations of the MPDE,
3. interpolate the solution onto a mesh with twice the number of intervals,
4. repeat steps 2–3 until we reach the required number of mesh intervals, and
5. then iterate until a stopping criterion is achieved.

This reduces the number of iterations required to $\mathcal{O}(\log_2 N)$, e.g., when $\varepsilon = 10^{-8}$ and $N = 1024$, iterations: $517 \rightarrow 27$ (3 on final mesh).

Efficient solution of MPDE

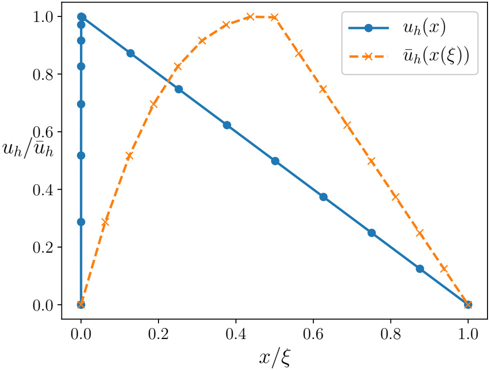
Example of how the mesh evolved when $N = 64$ and $\varepsilon = 10^{-3}$



Scalar 1D reaction-diffusion problem

$$-\varepsilon^2 u'' + u = 1 - x, \quad \text{on } (0,1), \quad u(0) = u(1) = 0.$$

Solution with $\varepsilon = 10^{-4}$ and $N = 16$ on the MPDE (physical) mesh ω^N and uniform (computational) mesh $\omega^{[c]}$



It can be shown (e.g., [Roos et al., 2008]) that the error in the linear-FEM solution generated on a Bakhvalov mesh satisfies

$$\|u - u^h\|_{\mathrm{E}} \leq C(\varepsilon^{1/2}N^{-1} + N^{-2}),$$

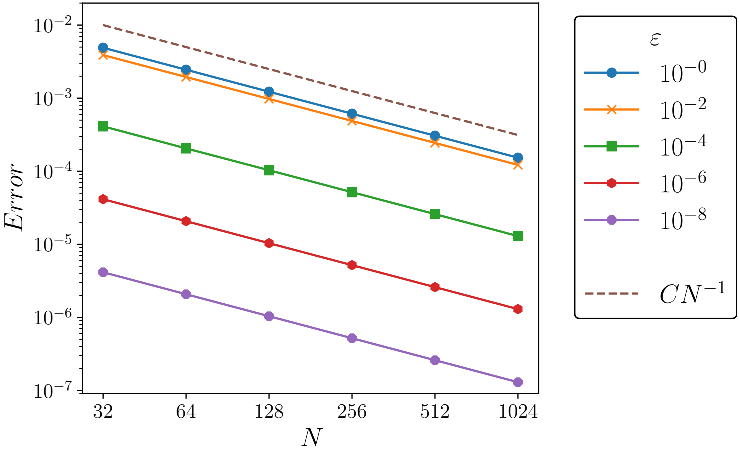
where u is the true solution and u^h is the FEM solution, C is a constant independent of ε and N, and $\|\cdot\|_{\mathrm{E}}$ is the usual energy norm induced by the FEM bilinear form.

In practice we compare the linear-FEM with the quadratic-FEM solution to compute our errors, e^h.

Scalar 1D reaction-diffusion problem

Errors

Plot of $\|e^h\|_{\mathrm{E}}$ for the scalar 1D problem when solved on a MPDE mesh, ω^N



2D example

Arguably, in 1D the method outlined has no advantage over other methods of equidistribution.

This would also be true in 2D if we restrict our interest to problems for which tensor product grids are appropriate.

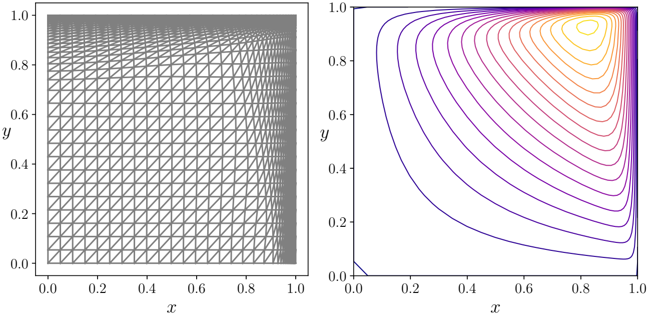
Therefore, we present a scenario where a non-tensor product grid is more appropriate.

The solution to the MPDE only determines the location of the mesh points, resulting in a unique mesh in 1D. However in 2D we also need connectivity, here our adapted-mesh inherits the connectivity from a uniform mesh.

2D problem with spatially varying diffusion

$$-\nabla \cdot \left(\begin{pmatrix} \varepsilon(1+2y)^2 & 0 \\ 0 & \varepsilon(3-2x)^2 \end{pmatrix}^2 \nabla u(x,y) \right) + u(x,y)$$
$$= (e^x - 1)(e^y - 1), \quad \text{for } (x,y) \in \Omega = (0,1)^2, \quad \text{with } u|_{\Omega} = 0.$$

MPDE Mesh and contour plot of solution when $\varepsilon = 10^{-2}$ and $N = 32$



2D problem with spatially varying diffusion
MPDE

The MPDE, for $\vec{x}(\xi_1,\xi_2) = (x,y)^T$, is

$$-\nabla \cdot (M(\vec{x}(\xi_1,\xi_2))\nabla\vec{x}(\xi_1,\xi_2)) = (0,0)^T, \text{ for } (\xi_1,\xi_2) \in \Omega^{[c]},$$

with boundary conditions,

$$x(0,\xi_2) = 0, \quad x(1,\xi_2) = 1, \quad \frac{\partial x}{\partial \mathrm{n}}(\xi_1,0) = 0, \quad \frac{\partial x}{\partial \mathrm{n}}(\xi_1,1) = 0,$$
$$y(\xi_1,0) = 0, \quad y(\xi_1,1) = 1, \quad \frac{\partial y}{\partial \mathrm{n}}(0,\xi_2) = 0, \quad \frac{\partial y}{\partial \mathrm{n}}(1,\xi_2) = 0,$$

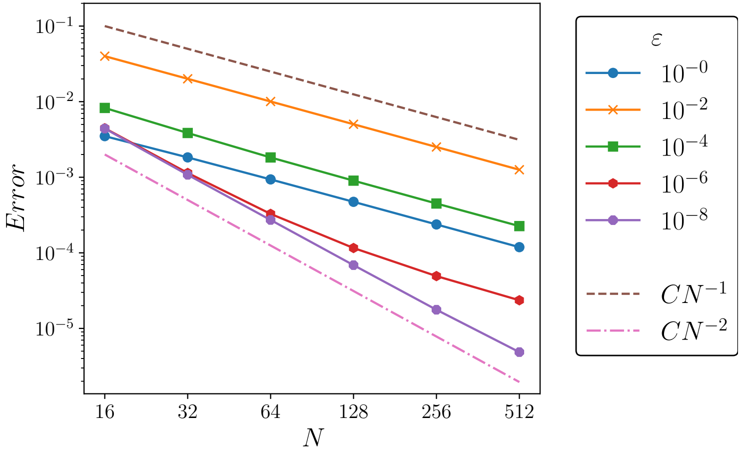
and

$$M(\vec{x}) = \begin{pmatrix} \max\left\{1, K_1 \frac{\beta}{\varepsilon(1+2y)^2} \exp\left(-\frac{\beta(1-x)}{\sigma\varepsilon(1+2y)^2}\right)\right\} & 0 \\ 0 & \max\left\{1, K_2 \frac{\beta}{\varepsilon(3-2x)^2} \exp\left(-\frac{\beta(1-y)}{\sigma\varepsilon(3-2x)^2}\right)\right\} \end{pmatrix}.$$

2D problem with spatially varying diffusion

Errors

Plot of $\|e^h\|_{\mathrm{E}}$



Conclusions

- The MPDE approach appears to be useful for generating suitable layer-adapted meshes, given sufficient *a priori* data.
- The method extends to 2D problems in situations where non-tensor product grids are appropriate.
- The MPDE is nonlinear and converges slowly for small ε and large N, we resolve this by combining the MPDE with *h*-refinement, the iteration count depends only very weakly on ε.
- For the 1D example provided, existing theory proves robust convergence. However, the 2D example provided does not have a theoretical basis.

Future work

- Our original motivation actually comes from solving some convection-diffusion-type problems: advection-diffusion and Navier Stokes. Initial results are promising.
- We are particularly interested in modelling dispersion and flow through constricted channels. MPDEs on convex sub-domains may be useful.
- We want to perform a comparison of alternative MPDE formulations.
- And, of course, we would like to incorporate *a posterori* error estimation. To date, best results are with hierarchical error estimators.

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#LoveIrish Research

From equidistribution to an MPDE

We derive an MPDE by considering the equidistribution principle as a mapping $x(\xi) : [0, 1] \to [a, b]$ from the computational coordinate ξ to the physical coordinate x, which satisfies

$$\int_a^{x(\xi)} \rho(x)dx = \xi \int_a^b \rho(x)dx. \quad (5)$$

Differentiating (5) twice with respect to ξ we get the nonlinear MPDE,

$$-(\rho(x)x(\xi)')' = 0 \text{ for all } \xi \in (0, 1), \quad x(0) = a \text{ and } x(1) = b. \quad (6)$$

The BCs are necessary to result in a mesh generating function.
The solution to (4) is a Bakhvalov mesh when ρ is as defined in (3).
Since this is a nonlinear problem we use a fixed point iteration method to find a solution.