Semismooth Newton method for Bingham flow

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1 Incompressible Fluids

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a bounded Lipschitz polyhedral domain and consider the system:

$$\begin{aligned} \alpha \boldsymbol{u} - \operatorname{div} \mathbf{S} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{f}, &\quad \Omega, \\ \operatorname{div} \boldsymbol{u} = 0, &\quad \Omega, \\ + \mathrm{BCs} & \end{aligned}$$

Here

- $\boldsymbol{u} \colon \Omega \to \mathbb{R}^d$ represents the velocity field;
- $p \colon \Omega \to \mathbb{R}$ is the pressure;
- $\mathbf{S} \colon \Omega \to \mathbb{R}^{d \times d}_{\mathrm{sym,tr}}$ is the shear stress tensor;

1 Constitutive relation (Bingham/Herschel–Bulkley)

Denote $\mathbf{D} := \mathbf{D}(\boldsymbol{u}) := \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)$.

$$\begin{cases} \mathbf{S} = 2\nu_*(|\mathbf{D}|)\mathbf{D} + \tau_* \frac{\mathbf{D}}{|\mathbf{D}|} & \text{if } |\mathbf{S}| \geq \tau_*, \\ \mathbf{D} = \mathbf{0} & \text{if } |\mathbf{S}| \leq \tau_*. \end{cases}$$

Here $\tau_* \geq 0$ is the yield stress and $\nu_* > 0$ is the viscosity.

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It can be naturally written using an implicit function:

$$\mathbf{G}(\mathbf{S}, \mathbf{D}) := (|\mathbf{S}| - \tau_*)^+ \mathbf{S} - 2\nu_*(\tau_* + (|\mathbf{S}| - \tau_*)^+)\mathbf{D} = \mathbf{0}.$$

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- M. Bulíček, P. Gwiazda, J. Málek, and A. Świerczewska-Gwiazda. *On unsteady flows of implicitly constituted incompressible fluids.* SIAM J. Math. Anal. 44(4):2756–2801, 2012.
- P.E. Farrell, P.A. Gazca-Orozco, and E. Süli. *Numerical analysis of unsteady implicitly constituted incompressible fluids: 3-field formulation.* SIAM J. Numer. Anal. 58(1):757–787, 2020.

1 Regularisation

A common regularisation [Bercovier, Engelman 1980]:

$$\mathbf{S}_\varepsilon = \tilde{\mathcal{S}}_\varepsilon(\mathbf{D}) := 2\nu_* \mathbf{D} + \tau_* \frac{\mathbf{D}}{\sqrt{\varepsilon^2 + |\mathbf{D}|^2}} \qquad \varepsilon > 0.$$

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The main alternative (AL) also has issues:

- In its basic form it can be slow.
- Needs more sophisticated tools;

1 New Regularisation

We employ here the simple regularisation from:

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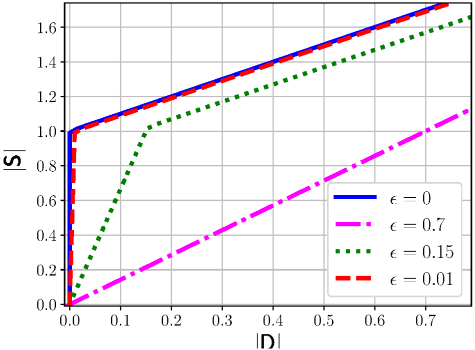

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- The stresses converge (weakly) $\mathbf{S}_\varepsilon \rightharpoonup \mathbf{S}$.
- The graph defined by $\mathbf{G}_\varepsilon$ is strongly monotone and 2-coercive:

$$\mathbf{S}_1 : \mathbf{D}_1 \geq c(|\mathbf{S}_1|^2 + |\mathbf{D}_1|^2) - \tilde{c},$$
$$(\mathbf{S}_1 - \mathbf{S}_2) : (\mathbf{D}_1 - \mathbf{D}_2) \geq c_\varepsilon(|\mathbf{S}_1 - \mathbf{S}_2|^2 + |\mathbf{D}_1 - \mathbf{D}_2|^2).$$

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The function $\mathbf{G}_\varepsilon$ is still not continuously differentiable.

We need a semismooth Newton method!

2 Semismooth Newton method

Classical Newton iteration for $F(z) = 0$:

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$$z^{k+1} = z^k - M_k^{-1}F(z^k).$$

Here M_k is an element of the generalised gradient of F, e.g. Clarke's differential (if $F\colon \mathbb{R}^m \to \mathbb{R}^n$):

$$\partial F(z) := \mathrm{co}\{M \in \mathbb{R}^{n\times m} \ : \ \exists\{z_i\} \subset \mathbb{R}^m \setminus U_R \text{ with } z_i \to z, \nabla F(z_i) \to M\}$$

Example

For $H(\mathbf{S}) = (|\mathbf{S}| - \tau_*)^+$ one has:

$$\partial H(\mathbf{S}) = \begin{cases} \{\mathbb{1}_{\{|\mathbf{S}|>\tau_*\}} \frac{\mathbf{S}}{|\mathbf{S}|}\} & \text{if } |\mathbf{S}| \neq \tau_*, \\ \{\phi \in \mathbb{R}^{d\times d} : |\phi| \leq 1\} & \text{if } |\mathbf{S}| = \tau_*. \end{cases}$$

For the positive part, UFL makes the choice:

$$\nabla \max\{f, 0\} = \begin{cases} \nabla f & \text{if } f > 0, \\ 0 & \text{if } f \leq 0. \end{cases}$$

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Proposition [Ulbrich,2003]

Suppose that in a nbd of the solution z we have $\|M^{-1}\|_{\mathcal{L}(X;Z)} \leq c$, and that

$$\sup_{M \in \partial F(z+h)} \|F(z+h) - F(z) - Mh\|_X = o(\|h\|_Z) \quad \text{as } h \to 0.$$

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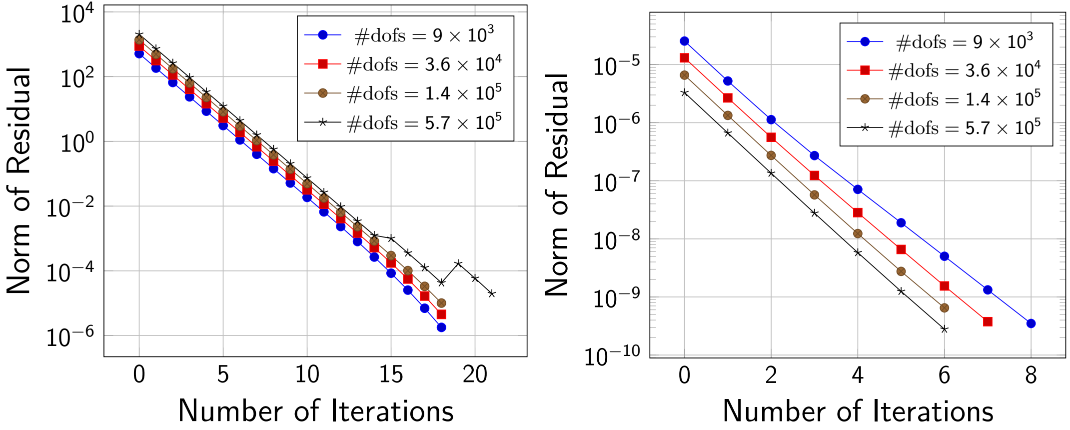
Need to carefully check that semismoothness of $\mathbf{G}\colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ implies that

$$(\mathbf{S}, \boldsymbol{u}) \in L^{r'}_{\mathrm{sym}}(\Omega)^{d\times d} \times W^{1,r}(\Omega)^d \mapsto \mathbf{G}(\mathbf{S}, \mathbf{D}(\boldsymbol{u})) \in L^{q}_{\mathrm{sym}}(\Omega)^{d\times d},$$

is semismooth.

2 Examples

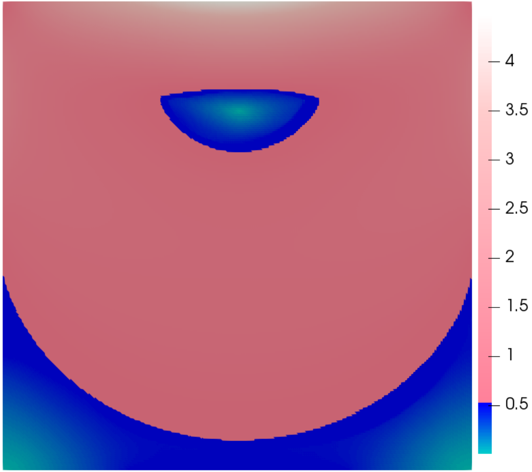
Using a stabilised $\mathbb{P}_0^{d\times d}$–$\mathbb{P}_1^d$–$\mathbb{P}_1$ element with firedrake:



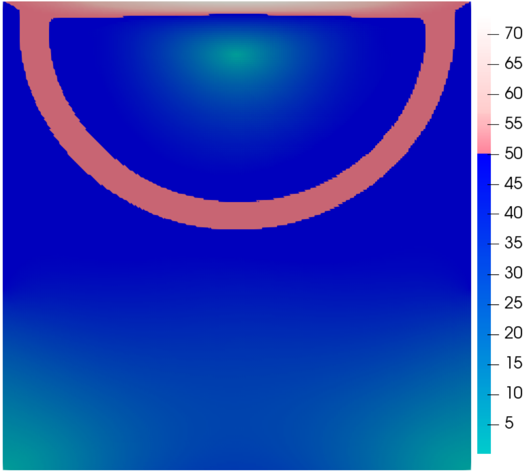
(a) $\varepsilon = 0.5$.

(b) $\varepsilon = 0.0001$.

2 Examples



(c) $\tau_* = 0.5$.



(d) $\tau_* = 50$.

Thank you for your attention!