Local a posteriori error estimates for the spectral fractional Laplacian

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March 24, 2021

I would like to acknowledge the support of the ASSIST research project of the University of Luxembourg. This presentation has been prepared in the framework of the DRIVEN project funded by the European Union's Horizon 2020 Research and Innovation programme

The spectral fractional Laplacian

- The spectral fractional Laplacian
- Contribution
- Discretization
- A posteriori error estimation
- Numerical results

The spectral fractional Laplacian

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The spectral fractional Laplacian

Let $\Omega \subset \mathbb{R}^d$, $\alpha \in (0,2)$ and $f \in L^2(\Omega)$.

$$(-\Delta)^{\alpha/2} u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{1}$$

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Let $\{\psi_i, \lambda_i\}_{i=1}^{+\infty} \subset L^2(\Omega) \times \mathbb{R}^+$ be such that

$$-\Delta \psi_i = \lambda_i \psi_i \quad \text{in } \Omega, \qquad \psi_i = 0 \quad \text{on } \partial\Omega, \quad \forall i = 1, \cdots, +\infty. \tag{2}$$

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The solution u of (1) is defined by

$$u := \sum_{i=1}^{+\infty} \lambda_i^{-\alpha/2} \left(f, \psi_i\right)_{L^2}. \tag{3}$$

Contribution

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Contribution

We present the first a posteriori error estimator for a numerical method presented in [Bonito and Pasciak, 2015] for solving the spectral fractional Laplacian.

Discretization

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Using an integral representation of the solution

$$u = C_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y} \, u_y \, \mathrm{d}y, \tag{4}$$

where u_y is solution to

$$\mathrm{e}^{2y} \int_\Omega \nabla u_y \cdot \nabla v + \int_\Omega u_y v = \int_\Omega f v, \quad \forall v \in H_0^1(\Omega). \tag{5}$$

Discretization

- Quadrature discretization: given a quadrature rule $\{\omega_l, y_l\}_{l=-N}^{N}$,

$$u = C_\alpha \int_{-\infty}^{+\infty} \mathrm{e}^{\alpha y}\, u_y \, \mathrm{d}y \approx C_\alpha \sum_{l=-N}^{N} \omega_l \, \mathrm{e}^{\alpha y_l}\, u_{y_l} =: u^N. \tag{6}$$

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- Finite element discretization: given a mesh $\mathcal{T}_h$ on Ω and V_h a FE space,

$$u \approx C_\alpha \sum_{l=-N}^{N} \omega_l \,\mathrm{e}^{\alpha y_l}\, u_{h,y_l} =: u_h^N, \tag{7}$$

where u_{h,y_l} solves

$$\mathrm{e}^{2y_l} \int_\Omega \nabla u_{h,y_l} \cdot \nabla v_h + \int_\Omega u_{h,y_l} v_h = \int_\Omega f v_h \quad \forall v_h \in V_h. \tag{8}$$

A posteriori error estimation

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We neglect the quadrature discretization error and we focus on the FE discretization error

$$\eta \approx \|u - u_h^N\|_{L^2}. \tag{9}$$

A posteriori error estimation

We consider the Bank–Weiser a posteriori error estimator [Bank and Weiser, 1985] on the parametric problem associated to u_{y_l}.

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$$\mathrm{e}^{2y} \int_T \nabla w_{T,y_l} \cdot \nabla v_T + \int_T w_{T,y_l} v_T = R_T(v_T) \quad \forall v_T \in V^{\mathrm{bw}}(T). \tag{10}$$

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The local fractional Bank–Weiser solution is given by

$$w_T := C_\alpha \sum_{l=-N}^{N} \omega_l \, e^{\alpha y_l} \, w_{T,y_l}. \tag{11}$$

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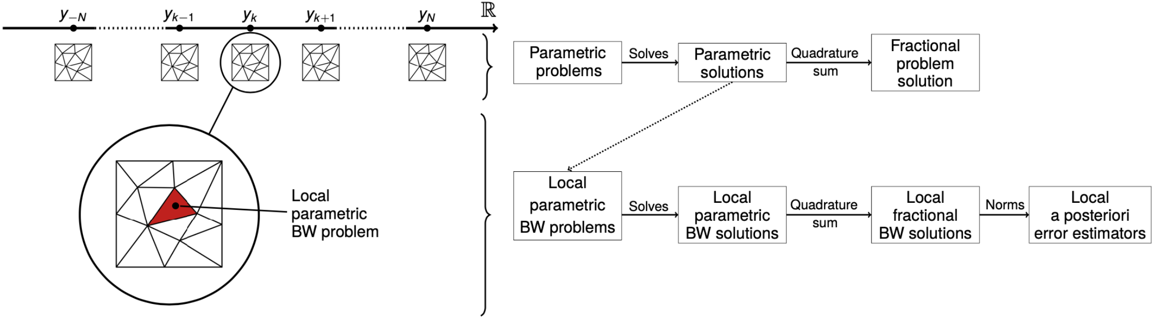
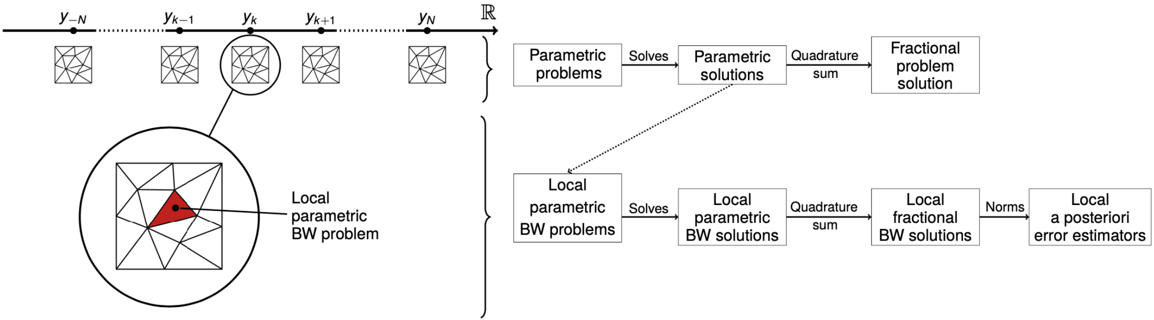
The local and global Bank–Weiser estimators are given by

$$\eta_{\mathrm{bw},T} := \|w_T\|_{L^2(T)}, \qquad \eta_{\mathrm{bw}}^2 := \sum_{T\in\mathcal{T}_h} \|w_T\|_{L^2(T)}^2. \tag{12}$$

A posteriori error estimation



A posteriori error estimation



Fully local and fully parallelizable.

Numerical results

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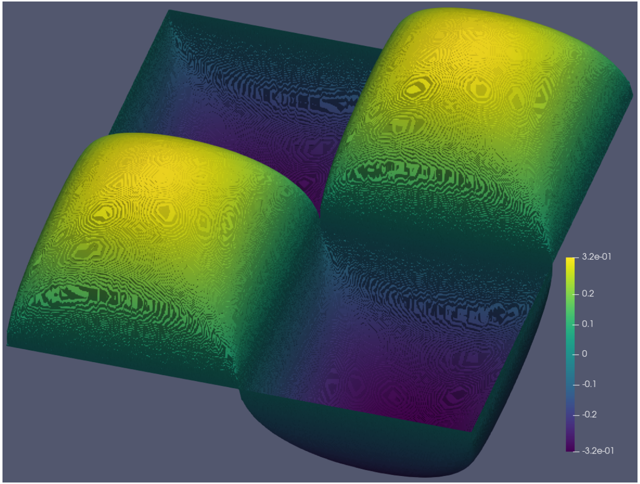
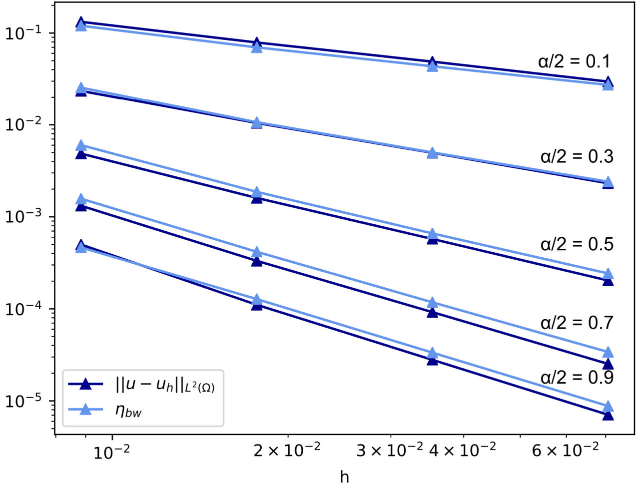
Numerical results



Numerical results

Uniform mesh refinement.



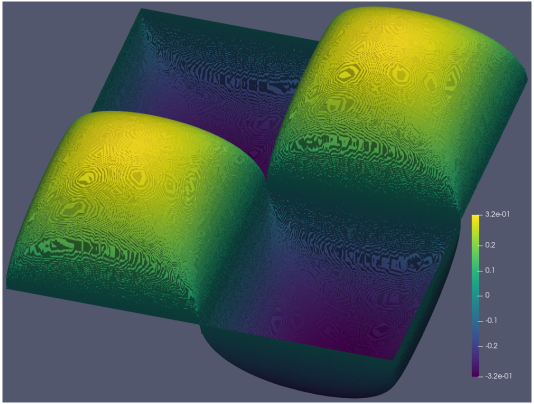
Numerical results

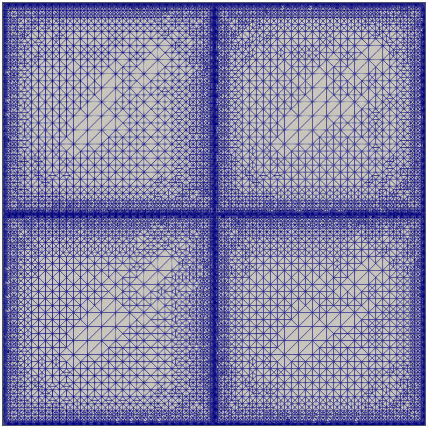
Uniform mesh refinement[Bonito and Pasciak, 2015].

Frac. pow.	0.1	0.3	0.5	0.7	0.9
Th. slope	0.7	1.1	1.5	1.9	2.0
Err. slope	0.71	1.11	1.52	1.9	2.04
Est. slope	0.71	1.13	1.54	1.84	1.91

Numerical results

Adaptive mesh refinement.





Numerical results

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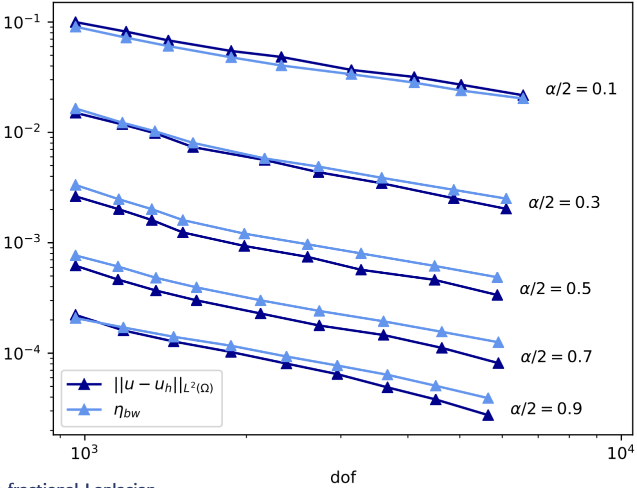

Numerical results

Adaptive mesh refinement.

Frac. pow.	0.1	0.3	0.5	0.7	0.9
Th. slope (unif.)	0.35	0.55	0.75	0.95	1.0
Err. slope (adapt.)	0.71	1.13	1.54	1.84	1.91
Est. slope (adapt.)	0.72	1.11	1.52	1.9	2.04

References I

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Thank you for your attention!

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I would like to acknowledge the support of the ASSIST research project of the University of Luxembourg. This presentation has been prepared in the framework of the DRIVEN project funded by the European Union's Horizon 2020 Research and Innovation programme under Grant Agreement No. 811099.