Piola-mapped finite elements for linear elasticity and Stokes flow

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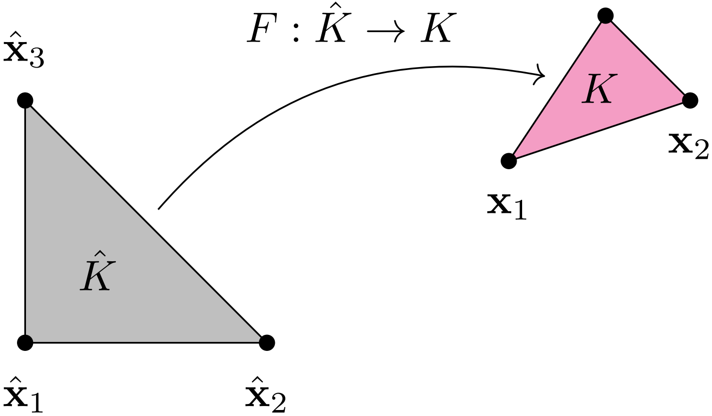
Motivation

The reference element

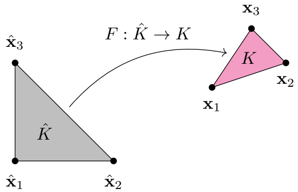


Motivation

The reference-to-physical map $F : K \to \hat{K}$ between cells

$$F(\hat{\mathbf{x}}) = J\hat{\mathbf{x}} + \mathbf{b}$$

induces a *pullback operator* on reference functions $\hat{\phi}$:



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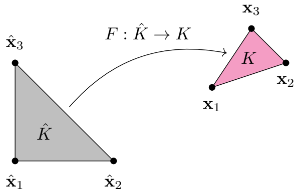

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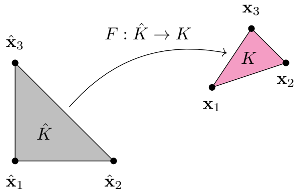

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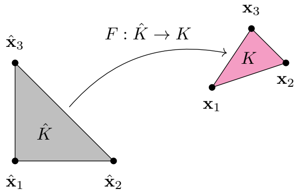

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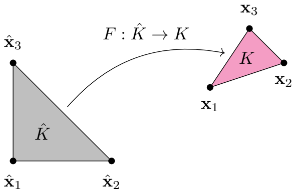

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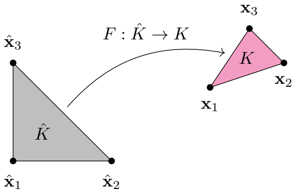

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Representative elements: $\begin{cases} H(\mathrm{div}) : & \text{Mardal–Tai–Winther [2002].} \\ H(\mathrm{div};\mathbb{S}) : & \text{Arnold–Winther [2002, 2003].} \end{cases}$

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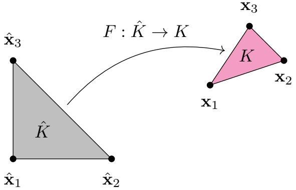

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$H(\mathrm{div}; \mathbb{S})$: **Arnold–Winther [2002, 2003].**

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$$\left(\hat{\Phi} : \hat{K} \to \mathbb{R}^d\right) \mapsto \left(\mathcal{F}^{\mathrm{div}}(\hat{\Phi}) = \Phi : K \to \mathbb{R}^d\right),$$
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Fact

These are isomorphisms

$$H(\text{div}, \hat{K}) \xrightarrow{\simeq} H(\text{div}, K), \qquad H(\text{div}, \hat{K}; \mathbb{S}) \xrightarrow{\simeq} H(\text{div}, K; \mathbb{S}).$$

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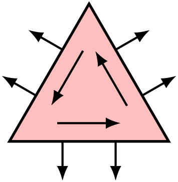
The Mardal–Tai–Winther element

$$MTW(K) = \left\{\Phi \in \mathcal{P}_3(K;\mathbb{R}^2) \,|\, \operatorname{div}\Phi \in \mathcal{P}_0(K), (\Phi\cdot\mathbf{n})\,|_{\mathbf{e}} \in \mathcal{P}_1(\mathbf{e})\ \forall \text{ edges } \mathbf{e}\right\}$$



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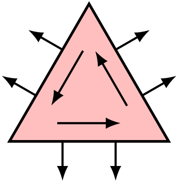

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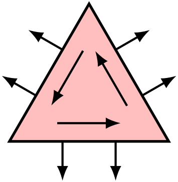

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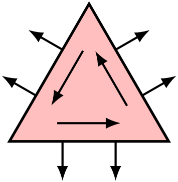

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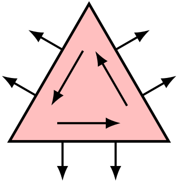

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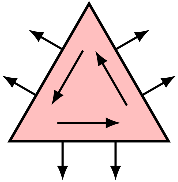

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- Discretises the 2D Stokes complex:

$$\mathbb{R} \xrightarrow{\subset} H^2(\Omega) \xrightarrow{\text{curl}} \mathbf{H}^1(\Omega) \xrightarrow{\text{div}} L^2(\Omega) \longrightarrow 0.$$

The Arnold–Winther stress elements

Conforming: $AW^c(K) = \{\tau \in \mathcal{P}_3(K;\mathbb{S}) \mid \operatorname{div}\tau \in \mathcal{P}_1(K;\mathbb{R}^2)\}$

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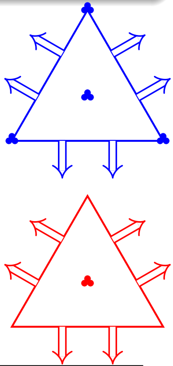

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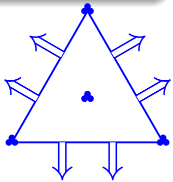

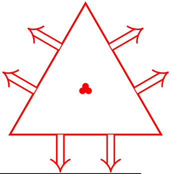

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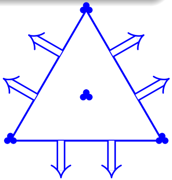

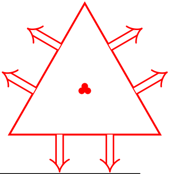

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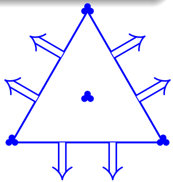
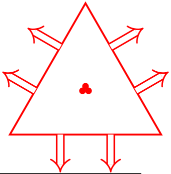
- **Almost never systematically implemented.**





Piola-inequivalent spaces

Denote:

- $\mathcal{F}^* : \hat{V} \to V$ a reference-to-physical Piola pullback
- $\hat{\Psi}, \Psi$ nodal bases for the MTW or AW spaces.

Then unfortunately,

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Proposition [A., Farrell, Kirby (2020)]

Explicit construction of the (sparse) dual transformations P for the MTW, AW^c, and AW^{nc} spaces.

Implementation: the MTW element

Implementation was carried out in **Firedrake** .

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A perturbed saddle point system

Seek $(u, p) \in (H_0(\mathrm{div}) \cap \epsilon \mathbf{H}^1_0(\Omega)) \times L^2(\Omega)$ such that

$$
\begin{aligned}
\left(I - \epsilon^2 \Delta\right) u - \nabla p &= f && \text{in } \Omega, \\
\operatorname{div} u &= g && \text{in } \Omega, \\
u &= h && \text{on } \Gamma_D, \\
\epsilon^2 \nabla u \mathbf{n} - p\mathbf{n} &= 0 && \text{on } \Gamma_N.
\end{aligned}
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$\epsilon = 1$: Stokes-like incompressible flow.
$\epsilon \to 0$: Darcy flow ($\sim$ mixed Poisson).

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\epsilon^2 \nabla u \mathbf{n} - p \mathbf{n} &= 0 && \text{on } \Gamma_N.
\end{aligned}$$

$\epsilon = 1$: Stokes-like incompressible flow.
$\epsilon \to 0$: Darcy flow ($\sim$ mixed Poisson).

We validate robustness with respect to ϵ using a smooth MMS on $\Omega = (0,1)^2$.

The MTW element: ϵ-independent MMS convergence rates

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$\epsilon \setminus N$	2^0	2^1	2^2	2^3	2^4	2^5	EOC
1	0.00456896	0.00128355	0.000341183	8.8577e-05	2.26311e-05	5.72514e-06	1.92807
2^{-2}	0.00447605	0.00126505	0.000335227	8.68331e-05	2.21707e-05	5.60825e-06	1.92809
2^{-4}	0.00421902	0.00120629	0.000319928	8.22134e-05	2.09019e-05	5.2806e-06	1.9284
2^{-6}	0.00405483	0.00113654	0.000305674	7.96031e-05	2.02428e-05	5.10031e-06	1.92697
2^{-8}	0.0040504	0.00112421	0.000296652	7.67045e-05	1.97471e-05	5.02906e-06	1.93071
2^{-10}	0.00405058	0.00112407	0.000296248	7.60249e-05	1.92762e-05	4.88509e-06	1.9391
0	0.00405059	0.00112407	0.000296246	7.60168e-05	1.92522e-05	4.84429e-06	1.94153

L^2 errors and convergence rates of MTW velocity for a range of ϵ.

$\epsilon \setminus N$	2^0	2^1	2^2	2^3	2^4	2^5	EOC
1	0.430242	0.200198	0.102848	0.051765	0.0259237	0.0129668	1.01045
2^{-2}	0.420754	0.198049	0.102341	0.051599	0.0258535	0.0129335	1.00476
2^{-4}	0.420711	0.19804	0.102339	0.0515984	0.0258533	0.0129334	1.00473
2^{-6}	0.420711	0.19804	0.102339	0.0515983	0.0258533	0.0129334	1.00473
2^{-8}	0.420711	0.19804	0.102339	0.0515983	0.0258533	0.0129334	1.00473
2^{-10}	0.420711	0.19804	0.102339	0.0515983	0.0258533	0.0129334	1.00473
0	0.420711	0.19804	0.102339	0.0515983	0.0258533	0.0129334	1.00473

L^2 pressure errors and convergence rates.

The Hellinger–Reissner principle

Seek a stress-displacement pair $(\sigma, u) \in H(\mathrm{div}; \mathbb{S}) \times \mathbf{L}^2(\Omega)$ such that

$$\begin{aligned} \mathcal{A}\sigma &= \varepsilon(u) && \text{in } \Omega, \\ \mathrm{div}\,\sigma &= f && \text{in } \Omega, \\ u &= u_0 && \text{on } \Gamma_D, \\ \sigma\mathbf{n} &= g && \text{on } \Gamma_N, \end{aligned}$$

where $\mathcal{A} = \mathcal{A}(\mu, \lambda)$ denotes the compliance tensor.

We validate the implementation again using a smooth MMS.

Arnold–Winther elements: MMS convergence rates

University of Oxford Mathematical Institute

N	u error	u EOC	σ error	σ EOC	$\mathrm{div}_h\ \sigma$ error	$\mathrm{div}_h\ \sigma$ EOC
2^1	0.10522	–	0.490015	–	0.832183	–
2^2	0.0278746	1.91638	0.209729	1.2243	0.267865	1.63539
2^3	0.00707398	1.97836	0.0890996	1.23504	0.0714705	1.90609
2^4	0.00177271	1.99656	0.0412454	1.11118	0.0181626	1.97638
2^5	0.000442977	2.00066	0.0199779	1.04583	0.00455929	1.99409

L^2 errors and convergence rates for AW^{nc}, at Poisson ratio $\nu = 0.3$.

N	u error	u EOC	σ error	σ EOC	$\mathrm{div}\ \sigma$ error	$\mathrm{div}\ \sigma$ EOC
2^0	4.94678	–	0.00816494	–	0.037794	–
2^1	1.0283	2.26623	0.0010553	2.95179	0.01005	1.91095
2^2	0.0826871	3.63646	0.000129246	3.02946	0.0025538	1.97649
2^3	0.00561489	3.88033	1.59231e-05	3.02093	0.000641093	1.99404
2^4	0.000365775	3.94023	1.98093e-06	3.00687	0.00016044	1.9985
2^5	2.7322e-05	3.74282	2.47482e-07	3.00078	4.01203e-05	1.99963

AW^c, near the incompressible limit $\nu = 0.49999$.

Arnold–Winther elements: Nitsche and smoothers

To enforce the traction condition

$$\sigma \mathbf{n} = g \qquad \text{on } \Gamma_N$$

which is **particularly difficult with AW elements** [Carstensen et al. (2008)],

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$$\mathcal{H}_{h,\gamma,\omega}(\sigma_h, u_h) := \int_\Omega \frac{1}{2}\mathcal{A}\sigma_h : \sigma_h + (\operatorname{div} \sigma_h - f) \cdot u_h$$
$$- \int_{\Gamma_N} (\sigma_h \mathbf{n} - g) \cdot u_h \ \mathrm{d}s + \frac{\gamma}{2h} \int_{\Gamma_N} \|\sigma_h \mathbf{n} - g\|^2 \ \mathrm{d}s \qquad .$$

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- Nitsche penalty for the traction condition,
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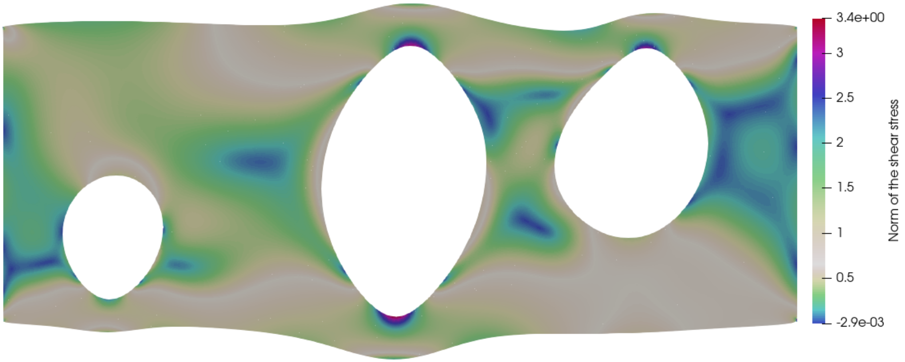
Patch-based additive Schwarz smoother [e.g. Schöberl (1999)]

We employ the *vertex-star iteration* to precondition the augmented stress block after block factorisation, applied by `PCASM`.



A traction-free condition except at both ends, coloured by the shear stress, near the incompressible limit ($\nu = 0.499999$); 1.14×10^6 DOFs using AW^{nc}. Considered in [Li (2018)].

Take-home messages

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- We have generalised contravariant Piola transformation theory to Piola-inequivalent elements.

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Thanks.